On Some Statistical Models in Life Testing

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Abstract

In this paper we consider a hazard rate of the form $h(t;\lambda)=kt+\varphi(t;\lambda)$, where $\varphi(t;\lambda)$ is (in most of the cases) a decreasing function as regards t. Various choices for k and φ provide interesting statistical models with applications in life testing and reliability. We will study the so called pseudo – Hjorth distribution function (nicknamed after the model proposed by the Swedish statistician Urban Hjorth in 1980), which has some interesting properties, as for instance the maximum of hazard rate being equal to the parameter distribution. This model can be also applied in the acceptance sampling, relating the fraction defective to the hazard rate.

1. Introduction

In an earlier paper (Orman, Bârsan-Pipu, Vodă 2002) we have presented the so-called *homographic hazard rate* of the form:

$$h(t)=\frac{a+bt}{c+dt}\,,\ a,b,c,d\in\mathbf{R}\,,\ t\geq 0\,, \tag{1}$$

and for a peculiar choice of the parameters – namely $a=0$, $d=\lambda>0$, $c=1$ and $b=\lambda^2$ – we obtain the following form:

$$h(t;\lambda)=\left(1-\frac{1}{1+\lambda t}\right)\cdot\lambda\,,\ \lambda>0\,,\ t\geq 0\,, \tag{2}$$

which is an increasing function with respect to the time (t).

The distribution function in this case is

$$T:F(t;\lambda)=1-(1+\lambda t)\cdot\exp(-\lambda t)\,,\ \lambda>0\,,\ t\geq 0\,, \tag{3}$$

and a main characteristic of such a variable (T) is that the variation coefficient, $CV(T)$, has a constant value $\left(\sqrt{2}/2\right)$, since $E(T)=2/\lambda$, and $Var(T)=2/\lambda^2$.

2. A general model for the hazard rate

In the present paper we consider a hazard rate of the form:

$$h(t;\lambda)=k\cdot t+\varphi(t;\lambda),\ t\geq 0\ k,\lambda\geq 0\,, \tag{4}$$

where $\varphi(t;\lambda)$ is (in most of the cases) a decreasing function as regards t.

Various choices for k and φ provide interesting statistical models with applications in life testing and reliability, as we discuss (proofs omitted) in the following cases.

2.1 A Rayleigh model

If $\varphi(t;\lambda)=\lambda t$, then $h(t;\lambda)=(k+\lambda)\cdot t$, which leads to the well-known Rayleigh model.

2.2 A generalized exponential model

If $k=0$ and $\varphi(t;\lambda)=1/\left(\lambda+e^{-t}\right)$, we see that $\lim_{t\to\infty}\varphi(t;\lambda)=1/\lambda$ and this describes the so-called "slow aging process". We have also $\frac{1}{1+\lambda}\le h(t;\lambda)\le\frac{1}{\lambda}$ and the distribution function is

$$T:F(t;\lambda)=1-\frac{\lambda+1}{\lambda+e^{t}}, \tag{5}$$

where for $\lambda=0$, we have $F(t;0)=1-e^{-t}$.

One can generalize (5) as follows:

$$F(t;\lambda,m)=1-\left(\frac{\lambda+m}{\lambda+me^{t}}\right)^{1/m},\ t\ge 0,\ m>0,\ \lambda\ge 0, \tag{6}$$

which for $\lambda=0$ becomes the classical exponential model

$T:F(t;0,m)=1-\exp(-t/m)$, with $E(T)=m$.

Therefore (6) may be considered as a *generalized exponential model*, different from that proposed by Khan in 1987 or that given by Vodă et al. in 1996 (see Bârsan-Pipu, Isaic-Maniu, Vodă, 1999, pp. 44-50).

2.3 An increasing model

If $k=1$ and $\varphi(t;\lambda)=\lambda/(1+t)$, we have

$$h(t;\lambda)=t+\frac{\lambda}{1+t},\ t\ge 0\ k,\lambda\ge 0, \tag{7}$$

which is an increasing function if $t\ge\sqrt{\lambda}-1$ ($\lambda>1$ is a mandatory condition).

2.4 A Burr-type model

If $k=0$ and $\varphi(t;\lambda)=\lambda/(1+t)$ – as in the above case – we obtain a special Burr-type distribution function

$$T:F(t;\lambda)=1-\frac{1}{(1+t)^{\lambda}}. \tag{8}$$

Remember that a Burr distribution function (proposed even in 1942) has the form

$$F(t;c,\lambda)=1-\left(1+t^{c}\right)^{-\lambda},\ t\ge 0,\ c,\lambda>0,$$

and in our case $c=1$. For details concerning the Burr function see Vodă (1982).

2.5 The pseudo - Hjorth model

If $k=0$ and $\varphi(t;\lambda)=2\lambda t/\left(1+t^{2}\right)$ we obtain

$$T:F(t;\lambda)=1-\frac{1}{\left(1+t^{2}\right)^{\lambda}}\ t\ge 0,\ \lambda>0. \tag{9}$$

The model parameter (λ) is just the maximum of the hazard rate function. We nicknamed this model as *pseudo - Hjorth model*. Urban Hjorth, the Swedish statistician, has proposed (see Hjorth, 1980) a distribution function of the form:

$$F(t;\theta,\delta)=1-\frac{e^{-\delta t^{2}/2}}{(1+t)^{\theta}},\ t\ge 0,\ \theta,\delta\ge 0, \tag{10}$$

where if $\theta=0$ one obtains also the Rayleigh model.

3. An application of the pseudo – Hjorth model in acceptance sampling

For the pseudo – Hjorth model we will present an application in acceptance sampling, relating the fraction defective (p) to the hazard rate. The values of p are taken as AQL (Acceptable Quality Level) preferential values given by ISO document #2859 or by the American standard MILSTD 105D "Sampling Procedures and Tables for Inspection by Attributes".

If the fraction defective is p, from (9) we may write

$$1-\frac{1}{\left(1+t^2\right)^{\lambda}}=p \quad \text{or} \quad \left(1-p\right)^{1/\lambda}=\frac{1}{1+t^2},$$

which provides

$$2\lambda t\cdot\left(1-p\right)^{1/\lambda}=\frac{2\lambda t}{1+t^2}. \tag{11}$$

The right side of (11) is just the hazard rate of the pseudo – Hjorth model since

$$f\left(t;\lambda\right)=2\lambda t\cdot\left(1+t^2\right)^{-(\lambda+1)} \quad \text{and hence} \quad h\left(t;\lambda\right)=\frac{f\left(t;\lambda\right)}{R\left(t;\lambda\right)}=\frac{2\lambda t}{1+t^2}.$$

Therefore, one may compute the ratio

$$\frac{h\left(t;\lambda\right)}{t}=2\lambda\cdot\left(1-p\right)^{1/\lambda}, \tag{12}$$

for various values of λ, taking the values of p as mentioned above.

As regards the estimation of λ, this can be easily done by maximum likelihood method as follows:

$$L\left(t_1,t_2,\ldots,t_n\right)=2^n\cdot\lambda^n\cdot\prod_{i=1}^{n}t_i\cdot\left(1+t_i^2\right)^{-(\lambda+1)},$$

$$\ln L=n\ln 2+n\ln\lambda+\sum_{i=1}^{n}\ln t_i-\left(\lambda+1\right)\cdot\sum_{i=1}^{n}\ln\left(1+t_i^2\right),$$

$$\frac{\partial\ln L}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\ln\left(1+t_i^2\right)=0.$$

which gives the ML estimates for $1/\lambda$ as

$$\widehat{\left(\frac{1}{\lambda}\right)}_{\mathrm{ML}}=\frac{1}{n}\cdot\sum_{i=1}^{n}\ln\left(1+t_i^2\right). \tag{13}$$

It is interesting to notice here that the variable $\ln\left(1+t^2\right)$ is exponentially distributed. Indeed, we may write

$$F\left(t_0\right)=\mathrm{Prob}\left\{\ln\left(1+t^2\right)<t_0\right\}=\mathrm{Prob}\left\{1+t^2<e^{t_0}\right\}=\mathrm{Prob}\left\{t<\left(e^{t_0}-1\right)^{1/2}\right\}.$$

Since t is pseudo – Hjorth distributed, we have from (9)

$$F\left(t_0;\lambda\right)=1-\frac{1}{\left[1+\left(e^{t_0}-1\right)\right]^{\lambda}}=1-e^{-\lambda\cdot t_0}. \tag{14}$$

Based on this property, we may find a lower tolerance limit $\Lambda_{\inf}$, that is:

$$\mathrm{Prob}\left\{\int_{\Lambda_{\inf}}^{+\infty}f\left(t;\lambda\right)dt\geq P\right\}=\gamma. \tag{15}$$

where the proportion P and the probability γ are given (for instance $P=0.95$ and $\gamma=0.90$).

We have:

$$\mathrm{Prob}\left\{\frac{1}{\left(1+\Lambda_{\inf}^{2}\right)^{\lambda}} \geq P\right\} = \gamma \quad \text{or} \quad \mathrm{Prob}\left\{\left(1+\Lambda_{\inf}^{2}\right)^{\lambda} \leq \frac{1}{P}\right\} = \gamma .$$

Taking logarithms, we obtain

$$\mathrm{Prob}\left\{\lambda \cdot \ln\left(1+\Lambda_{\inf}^{2}\right) \leq -\ln P\right\} = \gamma ,$$

and eventually one may use the estimate of λ from (13) in order to get the statistic $\Lambda_{\inf}$ (since the estimate of $1/\lambda$ is an average of n independent and identically distributed random variables, its distribution is approximately normal and hence we may use this fact to deduce $\Lambda_{\inf}$).

References

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